From random walks on networks to nonlinear diffusion

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Mathematical models of motility are often based on random-walk descriptions of discrete individuals that can move according to certain rules. It is usually the case that large masses concentrated in small regions of space have a great impact on collective movement of the group. For this reason, many models in mathematical biology have incorporated crowding effects and managed to understand their implications.

Here, we build on a previously developed framework for random walks on networks [1, 2] to show that in the continuum limit, the underlying stochastic process can be identified with a diffusion partial differential equation. The diffusion coefficient of the emerging equation is in general density-dependent, and can be directly related to the transition probabilities of the random walk.

In particular, the coarse-grained density of random walkers ρ is shown to satisfy the nonlinear diffusion equation

$$\partial_t \rho = \nabla \cdot (h(\rho)\nabla\rho)\,, \tag{1}$$

where the diffusion coefficient $h(\rho)$ is directly related to the transition probabilities of the microscopic random walk. With this we can recover many well-known models in mathematical biology.

Moreover, we can analytically characterize the stationary distribution and the relaxation time of the stochastic process on networks, which as usual, is linked to network structure, but also to the diffusion coefficient in Eq. (1). The employed argument can be generalised for modular networks, where one expects timescale separation. Here, we find that the nonlinear nature of the random walk also affects the different timescales of the process, as shown in Fig. 1.

Finally, these findings are applied to an analogue of a Fisher-KPP equation on networks. In the linear diffusion case, two regimes are observed depending on the timescales of diffusion and proliferation. We show that when diffusion is density-dependent, the same two regimes can be observed, although with important qualitative differences with respect to the linear diffusion setting.

This work is further developed in Ref. [3].

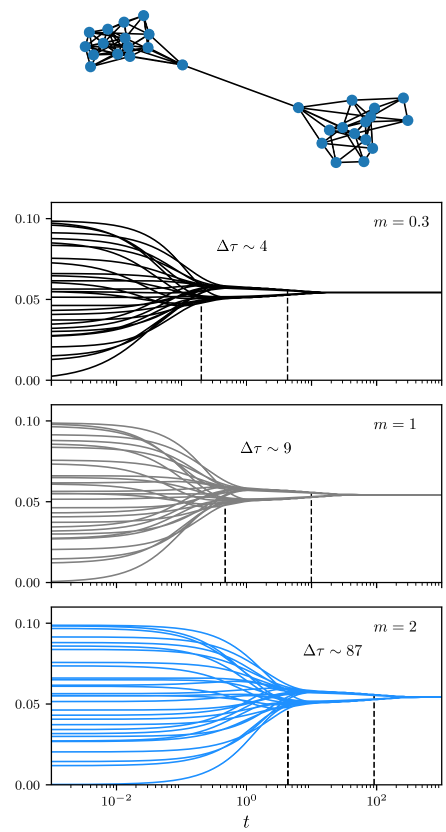

Fig. 1: Nonlinear diffusion on modular networks. Numerical simulations of $\mathrm{d}\rho/\mathrm{d}t = -L\rho^m$ on the shown graph. Dashed lines represent the timescales that we calculate analytically.

References

[1] Malbor Asllani, Timoteo Carletti, Francesca Di Patti, Duccio Fanelli, and Francesco Piazza. Hopping in the crowd to unveil network topology. *Phys. Rev. Lett.*, 120:158301, Apr 2018.

[2] Timoteo Carletti, Malbor Asllani, Duccio Fanelli, and Vito Latora. Nonlinear walkers and efficient exploration of congested networks. *Phys. Rev. Research*, 2:033012, Jul 2020.

[3] Carles Falcó. From random walks on networks to nonlinear diffusion. *arXiv preprint arXiv:2206.08859*, 2022.