Perturbation-based graph theory: an integrative dynamical perspective for the study of complex networks

Gorka Zamora-López[1,*], and Matthieu Gilson[2,†]
[1]Center for Brain and Cognition, Universitat Pompeu Fabra, Barcelona, Spain.
[2]Institut des Neurosciences des Systemes, INSERM-AMU, Marseille, France.

Built upon the shoulders of graph theory, the field of complex networks has become a central tool for understanding complex systems. Represented as a graph, empirical systems across domains can thus be studied using the same concepts and the same metrics. However, this simplicity is also a major limitation since graph theory is defined for a binary and symmetric description where the only relevant information is whether a link exists or not between two vertices. Despite the successful adaptation of graph theory to directed graphs, its application to weighted networks has been rather clumsy. Empirical relations are usually weighted and we daily face the need to take arbitrary choices like, for example, having to threshold the real data to obtain a binary matrix on which, now yes, the graph tools can be applied.

Here, we propose a reformulation of graph theory from a dynamical point of view that can help aleviate these limitations, valid at least for the class of real networks that accept flows. First, we show that classical graph metrics are derived from a simple but common generative dynamical model (a discrete cascade) governing how perturbations propagate along the network. The Green's function $\mathcal{C}(A,t)$ of the adjacency matrix A for the discrete cascade represents the network response to unit external perturbations at consecutive discrete times t. All the relevant information needed to describe the network, and to define graph metrics, is unfolded via the generative dynamics from the adjaceny matrix A onto its Green's function $\mathcal{C}(t)$, see Fig. 1A. From this perspective, graph metrics are no longer regarded as combinatorial attributes of a graph A, but they correspond to spatio-temporal properties of the network's response to external perturbations.

Second, seen from this dynamical angle, we learn that the difficulties of graph theory to deal with weighted networks are the consequence of the constrains of its "hidden" dynamical model, rather than a limitation imposed by the binary representation. Therefore replacing the underlying discrete cascade by other generative models (either discrete or continuous, conservative or non-conservative) network metrics can be redefined from the corresponding Green's function $\mathcal{C}(t)$ of each model. For example, graph distance is typically evaluated as the minimal number of "hops" needed to traverse between two nodes. But in the case of weighted networks "hops" is no longer a valid metric of distance. Instead, from a dynamical point of view, the time that a perturbation on node i takes to significantly affect other vertices j can be used to redefine their distance, as shown in Fig. 1B. Another limitation of graph theory is the difficulty for comparing across networks. Under this framework, a simple renormalization of the connectivity matrices allows to align networks of same architecture but of different densities or sizes as shown in Fig. 1C.

In summary, we propose a dynamical formulation of graph theory in which the underlying generative model is explicit and tunable. This allows to define metrics in which both directionality and link weights are natural – built-in – aspects of the metrics. This flexibility provides the oportunity to calibrate network analyses by choosing generative models that are better suited for the specific system under study; thus balancing between simplicity and interpretability of results. A plethora of past efforts have employed different types of dynamics to study and characterise complex networks, e.g., by navigating on them [1], the propagation of random walkers [2] or via routing models [3]. We envision that the perturbative formulation here proposed serves to enclose all those efforts under a common umbrella.

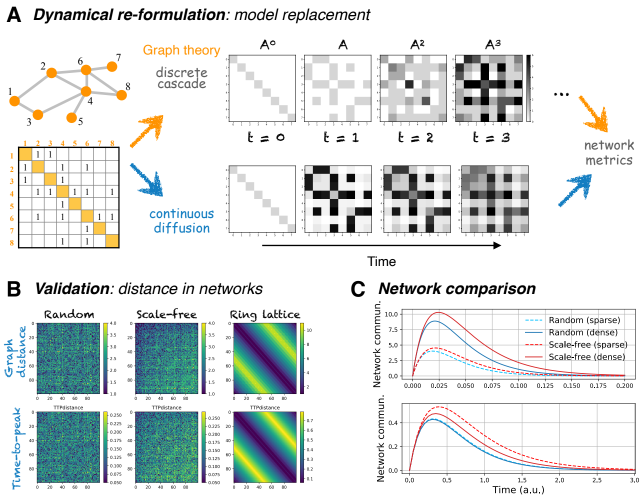

Fig. 1. Graph metrics emerge from a common hidden generative model – a discrete and non-conservative cascade. The powers of the adjacency matrix describe the network response (effect between nodes) over time due to initial unit perturbations, encoding all necessary information to characterise the graph and define metrics. Replacing the generative model allows to derive generalised graph metrics in which directionality and link weights are naturally encompased.

* *gorka@zamora-lopez.xyz* https://www.zamora-lopez.xyz
† *matthieu.gilson@univ-amu.fr* https://matthieugilson.eu

[1] M. Boguña, D. Krioukov and K.C. Claffy, *Navigability of complex networks*, Nat. Phys. **5**, 74 (2008).

[2] R. Lambiotte, J.C. Delvenne and M. Barahona, *Random Walks, Markov Processes and the Multiscale Modular Organization of Complex Networks*, IEEE Trans. Net. Sci. Eng. **1(2)** 76 (2014).

[3] A. Muscolini, J.M. Thomas, S. Ciucci, G. Bianconi and C.V. Canistracci, *Machine learning meets complex networks via coalescent embedding in the hyperbolic space*, Nat Comms. **8**:1615 (2018).