Topological Stability of Simplicial Complexes through Matrix ODE

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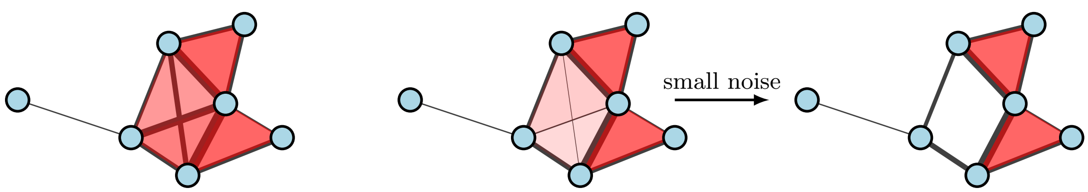
Motivation. The homology groups of a simplicial complex reveal fundamental properties about the topology of the data or the system. However, little is known about the *stability* of these groups. The question of how close is a given simplicial complex $\mathcal{K}$ to one with different homology is both interesting and challenging. For example, if $\mathcal{K}$ is formed as a result of some physical measurement (e.g. fMRI, chemical reactions, etc.), small errors in the data collection process may result in very different homology groups. At the same time, if we are given a complex $\mathcal{K}$ affected by noise, we expect that a small change to the observed $\mathcal{K}$ is enough to recover the actual underlying topology. These examples can be formulated as a problem of stability (or sensitivity): given a simplicial complex $\mathcal{K}$, what is and how large is the smallest perturbation to $\mathcal{K}$ that changes its homology groups?



(a) Stable simplicial complex (b) Unstable complex close to one with different homology

Figure 1: Example of the stable and unstable simplicial complexes; triangles in the simplicial complexes are colored in red, weight of the edges shown by width.

Problem formulation. In this work we address this question from the numerical point of view: Given a simplex $\mathcal{K}$, we develop an algorithm to compute (an approximation of) the smallest perturbation of the edge set of $\mathcal{K}$ (so, the set of its 1-simplicies) so that the Hodge homology of $\mathcal{K}$ increases. As it is the most common in applications and since it provides already a large number of numerical complications, we focus on the Hodge homology here, defined as the dimension of the kernel of the Hodge Laplacian $L_1 = B_0^\top B_0 + B_1 B_1^\top$, where B_k is the k-th complex's boundary operator.

Our approach is based on a perturbative analysis model: we consider a matrix differential equation (a gradient system) for the Hodge Laplacian L_1 which continuously models the variation of the edge weights of $\mathcal{K}$ and simultaneously minimizes a cost functional that aims at growing the kernel of L_1.

In order to deal with continuous weight perturbations, we consider a dynamics on the normalized version of the k-th boundary operator

$$\overline{B}_k(t) = W_{k-1}^{-1}(t) B_k W_k(t)$$

where the $W_k(t)$ are diagonal matrices of the weights of the k-simplicies, which change throughout the dynamics. This continuous perturbation of the boundary operator introduces a number of issues which we have to carefully deal with, including so-called homological pollution — noise in the kernel of L_1 due to higher or lower order homologies — and the generation of faux edges, i.e. the presence of all-zero rows and columns in L_1.

Numerical Optimization and Results. We evolve $\overline{B}_k(t)$ using a two-phase scheme: on the inner level, we find the optimal perturbation δW_1 for a fixed norm $\|\delta W_1\| = \varepsilon > 0$ of possible changes in the edge weights; on the outer level, we change the magnitude ε of the perturbation via a homotopical transition. We illustrate the behavior of the algorithm on a number of synthetic and real-world example simplicies.