Abstract

We are interested in the Lie algebraic structure of colored network dynamical systems, as in:

Example: A colored network dynamical system with two colors ($C = 2$) and four cells ($N = 4$):

\[
\begin{align*}
\dot{x}_1 &= f^1(x_1, x_2; x_3, x_4), \\
\dot{x}_2 &= f^1(x_2, x_1; x_4, x_3), \\
\dot{x}_3 &= f^2(x_3, x_4; x_1, x_2), \\
\dot{x}_4 &= f^2(x_4, x_3; x_2, x_1).
\end{align*}
\]

In the classification of the normal form of a colored network vector field, following the semigroup(oid) approach in [3], one would like to be able to say something about the structure of the Lie algebra of the linear colored networks. We describe in [2] a concrete algorithm (cf. [1]) that gives us the block decomposition for the Lie algebra of the linear part ($\text{net}_{C,N}$) as a block matrix with $B = N - C$,

\[
\text{net}_{C,N} = \begin{bmatrix}
\begin{array}{cccc}
  c_{11} & \cdots & c_{1C} \\
  \vdots & \ddots & \vdots \\
  c_{C1} & \cdots & c_{CC} \\
  a_1^{11} & \cdots & a_1^{1B} \\
  \vdots & \ddots & \vdots \\
  a_C^{11} & \cdots & a_C^{1B} \\
  b_1^{11} & \cdots & b_1^{1B} \\
  \vdots & \ddots & \vdots \\
  b_C^{11} & \cdots & b_C^{1B}
\end{array}
\end{bmatrix}
\begin{bmatrix}
  0 & \cdots & 0 \\
  \vdots & \ddots & \vdots \\
  0 & \cdots & 0 \\
  a_1^{11} & b_1^{11} & \cdots & b_1^{1B} \\
  \vdots & \ddots & \vdots \\
  a_C^{11} & b_C^{11} & \cdots & b_C^{1B}
\end{bmatrix}
\] = \begin{bmatrix}
c & 0 \\
a & b
\end{bmatrix} = \begin{bmatrix}
\mathfrak{gl}_C & 0 \\
\mathfrak{gl}_B & \mathfrak{gl}_B
\end{bmatrix}.
\]

We show that for $N$-dimensional vector fields with $C$ colors (different functions describing different types of cells in the network) this Lie algebra $\text{net}_{C,N}$ is isomorphic to the semidirect sum of a semisimple part, consisting of two simple components $\mathfrak{sl}_C$ and $\mathfrak{sl}_B$, with, which we write as a block-matrix and a solvable part, consisting of two elements representing the identity $\mathfrak{c}$ in $\mathfrak{c} \simeq \mathfrak{gl}_C$ and $\mathfrak{b}$ in $\mathfrak{b} \simeq \mathfrak{gl}_B$, and an abelian algebra $\mathfrak{a} \simeq \mathfrak{Gr}(C, N)$, the Grassmannian, consisting of the $C$-dimensional subspaces of $\mathbb{R}^N$.

References

