Balanced Hodge Laplacians Optimize Consensus Dynamics over Simplicial Complexes

Cameron Ziegler[1] (cz22@buffalo.edu), Per Sebastian Skardal[2] (persebastian.skardal@trincoll.edu), Haimonti Dutta[3] (haimonti@buffalo.edu), and Dane Taylor[1] (danet@buffalo.edu)

[1] Department of Mathematics, University at Buffalo, State University of New York
[2]Department of Mathematics, Trinity College
[3]Management Science and Systems Department, University at Buffalo, State University of New York

Despite the vast literature on network dynamics, we still lack basic insights into dynamics on higher-order structures (e.g., edges, triangles, and more generally, k-dimensional "simplices") and how they are influenced through higher-order interactions. A prime example lies in neuroscience where groups of neurons (not individual ones) may provide the building blocks for neurocomputation. Here, we study consensus dynamics on edges in simplicial complexes using a type of Laplacian matrix called a Hodge Laplacian, which we generalize to allow higher- and lower-order interactions to have different strengths. Using techniques from algebraic topology, we study how collective dynamics converge to a low-dimensional subspace that corresponds to the homology space of the simplicial complex. We use the Hodge decomposition to show that higher- and lower-order interactions can be optimally balanced to maximally accelerate convergence, and that this optimum coincides with a balancing of dynamics on the curl and gradient subspaces. We additionally explore the effects of network topology, finding that consensus over edges is accelerated when 2-simplices are well dispersed, as opposed to clustered together. [1]

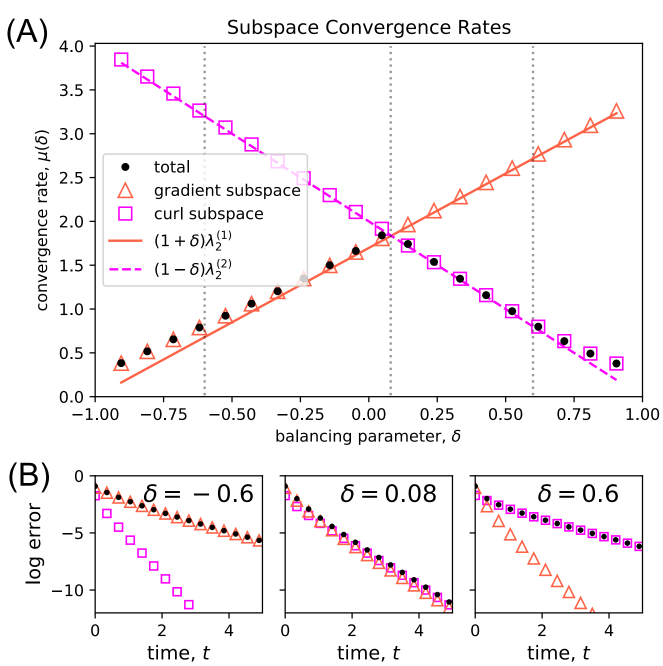

Figure 1: **Balancing invariant subspace dynamics maximally accelerates convergence**. Working with a specific simplicial complex, plot (A) shows the convergence rate of the total value as well as the gradient and curl subspaces in Generalized Hodge Laplacian-1 consensus simulations for different values of the balancing parameter, δ. The orange and purple lines show the expected values for these convergence rates calculated using the eigenvalues of the component matrices of the generalized Hodge Laplacian corresponding to the gradient and curl subspaces. For this simplicial complex, the balancing parameter that optimizes convergence rate is $\delta^* \approx .08$. In (B), we plot the log normed error, $\log(||\mathbf{x}(t) - \mathbf{x}^{(\mathbf{h})}||_2)$, again for the total as well as the gradient and curl subspaces for a few select values of δ.

[1] C. Ziegler, P. S. Skardal, H. Dutta, and D. Taylor, arXiv preprint arXiv:2112.01070 (2021).