Hypergraphs or simplicial complexes: Untangling the effect of higher-order representations on collective dynamics

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For the past three decades, networks have been successfully used to model complex systems with many interacting units. In their traditional form, networks only encode pairwise interactions. Growing evidence, however, suggests that a node may often experience the influence of multiple other nodes in a nonlinear way, and that such higher-order interactions cannot be decomposed into pairwise ones [1, 2, 3, 4]. Examples can be found in a wide variety of domains including human dynamics, collaborations, ecological systems, and the brain. Higher-order interactions not only impact the structure of these systems, they also often reshape their collective dynamics. Indeed, they have been shown to induce novel collective phenomena, such as explosive transitions, in a variety of dynamical processes including diffusion, consensus, spreading, and evolution.

Despite many recent theoretical advances, little attention has so far been given to how higher-order interactions are best represented. There are two mathematical frameworks that are most commonly used to model systems with higher-order interactions: hypergraphs and simplicial complexes. In most cases, the two representations have been used interchangeably and the choice for one or the other often appears to be motivated by technical convenience. For example, topological data analysis and Hodge decomposition require simplicial complexes. Here, we ask: Are there hidden consequences of choosing one higher-order representation over the other that could significantly impact the collective dynamics?

To answer this question, we focus on synchronization, a paradigmatic process for the emergence of order in populations of interacting entities. It underlies the function of many natural and man-made systems, from circadian rhythms and vascular networks to the brain. Nonpairwise interactions arise naturally in synchronization from the phase reduction of coupled oscillator populations [5, 6]. A key question regarding higher-order interactions in this context is: When do they promote synchronization? Recently, hyperedge-enhanced synchronization has been observed for a range of node dynamics [7, 8, 9]. It is thus tempting to conjecture that nonpairwise interactions synchronize oscillators more efficiently than pairwise ones. This seems physically plausible given that higher-order interactions enable more nodes to exchange information simultaneously, thus allowing more efficient communication and ultimately leading to enhanced synchronization performance.

In this talk, we will show that whether higher-order interactions promote or impede synchronization is highly representation-dependent. In particular, through a rich-get-richer effect, higher-order interactions consistently destabilize synchronization in simplicial complexes. On the other hand, through a homogenizing mechanism, higher-order interactions tend to stabilize synchronization in random hypergraphs. We further link the opposite trends to the different higher-order degree heterogeneities under the two representations, offering a theoretical underpinning for the representation-dependent synchronization performance. Since degree heterogeneity plays a key role not only in synchronization, but also in other dynamical processes such as diffusion and contagion, the effect of higher-order representations discovered here is likely to be crucial in complex systems beyond coupled oscillators.

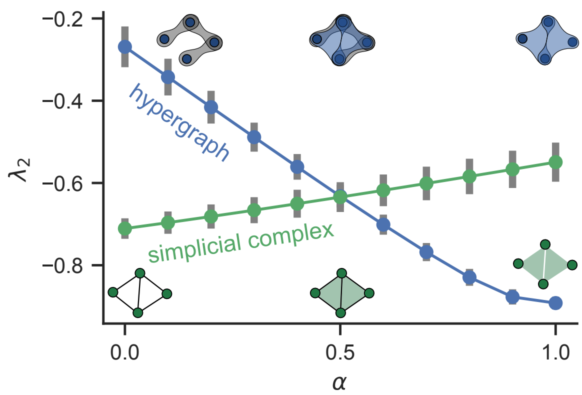

Fig. 1. **Synchronization is enhanced by higher-order interactions in random hypergraphs but is impeded in simplicial complexes.** The maximum transverse Lyapunov exponent λ_2 is plotted against α for random hypergraphs (blue) and simplicial complexes (green). As α is increased, the coupling goes from first-order-only ($\alpha = 0$) to second-order-only ($\alpha = 1$). Each point represents the average over 50 independent hypergraphs or simplicial complexes with $n = 100$ nodes. The error bars represent standard deviations. We set the connection probabilities $p = p_\triangle = 0.1$ for random hypergraphs and $p = 0.5$ for simplicial complexes.

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