Community structure in hypergraphs and the emergence of polarization

Nicholas W. Landry[1,2] and Juan G. Restrepo[2]
[1]Complex Systems Center, University of Vermont, Burlington, Vermont, USA
[2]Department of Applied Mathematics, University of Colorado at Boulder, Boulder, Colorado, USA

Society has become increasingly divided in politics, social media, and ideology. Researchers have found that polarization is often the result of "echo chambers" where a community reinforces its own beliefs while another community does the same with a different set of beliefs. Many empirical complex systems include group interactions and we find that a simple model of social contagion can give rise to polarization when these group interactions are included for strong enough community structure.

Hypergraphs are mathematical objects that encode relationships between an arbitrary number of entities, or nodes and an interaction involving m nodes is known as a m-hyperedge. An *m-uniform hypergraph* is a hypergraph that only contains hyperedges of size m. The mth-order degree of a node i, $k_i^{(m)}$ is the number of m-hyperedges to which node i belongs.

We model the propagation of an ideology with the hypergraph susceptible–infected–susceptible (SIS) model, where the ideology (represented with a binary variable) can spread via pairwise or three-way interactions through the collective contagion process [1]. In this model, an infected node spontaneously transitions to the susceptible state at a rate γ and a susceptible node transitions to the infected state at a rate β_m if that node is a member of a group of size m where all other members are infected. We comment that that this model contains asymmetry; the contagion will die out on its own and needs group pressure to sustain it.

We model community structure with the *planted partition model* for m-uniform hypergraphs. This model specifies that there are two equally-sized communities, the mean degree of the hypergraph remains fixed, and that a single *imbalance parameter* ϵ_m controls the community structure [2]. We say that a hyperedge is *intra-community* if all its members have the same group label and *inter-community* otherwise. When $\epsilon_m = 0$, this model specifies an Erdös-Rényi m-uniform hypergraph, and when $\epsilon_m = \langle k^{(m)}\rangle$, we obtain an m-uniform hypergraph with two completely disconnected communities.

Assuming independence of node states, we can approximate the contagion dynamics on a hypergraph constructed from both the 2-uniform and 3-uniform planted partition model with a two-variable mean-field equation

$$\frac{dx_1}{dt} = -\gamma x_1 + \frac{\beta_2}{2}(1-x_1)\left[\langle k^{(2)}\rangle(x_1+x_2) + \epsilon_2(x_1-x_2)\right] + \frac{\beta_3}{4}(1-x_1)\left[\langle k^{(3)}\rangle(x_1+x_2)^2 + \epsilon_3(3x_1^2 - 2x_1x_2 - x_2^2)\right], \tag{1}$$

where x_1 and x_2 are the fraction of nodes infected in communities 1 and 2 respectively. Note that the equation for $\frac{dx_2}{dt}$ can be obtained by substituting $x_1 \longleftrightarrow x_2$.

Assuming that $x_1 = x_2$, we find that $\beta_2^c/\gamma = 1/\langle k^{(2)}\rangle$ determines the epidemic threshold and that $\beta_3^c/\gamma = 1/\langle k^{(3)}\rangle$ determines the onset of bistable behavior. From these expressions, we can see that neither the epidemic threshold nor the onset of bistability depend on the community structure. We rescale the mean-field equations by setting $\gamma = 1$ and defining $\widetilde{\beta}_m = \beta_m/\beta_m^c$ and $\widetilde{\epsilon}_m = \epsilon_m/\langle k^{(m)}\rangle$ so they are defined as fractions of their critical values. With strong enough community structure, we see the emergence of stable asymmetric fixed points, where two communities may hold very different average opinions. We define the maximum distance between their stable fixed points as the *polarization* and in Fig. 1, we show the 2- and 3-hyperedge community structure for which polarization is possible.

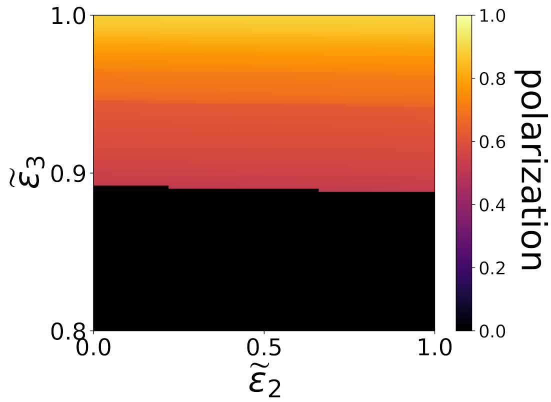

Fig. 1. The polarization computed from the mean-field equations.

We find that the existence of polarization is most strongly influenced by the community structure of the 3-hyperedges and that the structure of the 2-hyperedges has almost no effect on the existence of polarization.

Lastly, we simulate the stochastic contagion model on a synthetic dataset and the United States Congress bill cosponsorship dataset filtered to only include 2- and 3-hyperedges. The synthetic dataset validates our mean-field predictions and the cosponsorship dataset reveals that not only does polarization occur in empirical datasets, but that heterogeneity in the degree distribution has a strong effect on the magnitude of the polarization.

[1] N.W. Landry and J.G. Restrepo, *The Effect of Heterogeneity on Hypergraph Contagion Models*, Chaos **30**, 103117 (2020).

[2] M. Jerrum and G.B.. Sorkin, *The Metropolis Algorithm for Graph Bisection*, Disc. App. Math. **82**, 155–175 (1998).