Directed percolation in temporal networks

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Percolation theory forms the theoretical backbone of our understanding of the connectivity and reachability in networks. Isotropic percolation theory enables theoretically grounded ways to characterise and study phase transitions such systems go through. The critical phenomena observed in percolation have direct physical interpretations, such as the percolation threshold equating to the epidemic threshold and cluster sizes equating to final epidemic sizes.

While research on network connectivity and spreading processes, most recently network epidemiology, has exploded in popularity, the theoretical network-based work is still largely based on static network analysis, which is a grossly simplified model of how interactions take place with major implications to connectivity. It has been shown before that the temporal inhomogeneities and the specific order of event has real effects on rapidity and reach of spreading processes [4]. The percolation theory, however, has not yet been extended to temporal networks, leaving us without an understanding of connectivity and related dynamical phenomena in a same way as it has been established on static networks.

We tackle this fundamental problem and show that temporal network reachability as induced by the limited waiting-time process, a generic notion of constrained connectivity, is in the *directed percolation* universality class. We employ the theoretical advances in temporal event graphs [5] to define the appropriate set of quantities that aptly describe the state of the system and derive the critical exponents of percolation under a mean-field assumption of connectivity [3]. Furthermore, we use a novel probabilistic counting algorithm to estimate reachability in large temporal networks [1], similar to the example shown in Fig. 1, to confirm our analytical result on a broader set of networks with various topological and temporal inhomogeneities via extensive simulations [2]. Both analytically and empirically, we find that the connectivity in temporal networks can be explained by a set of critical exponents that are the same as the critical exponents of any system in the directed percolation universality class. Furthermore, we show that many real-world temporal networks, from human communication and social networks to transportation networks, display one or more similar phase transitions in reachability.

This work aims to provide the missing link between the well-grounded study of directed percolation to spreading processes on temporal networks, by providing a set of characteristic quantities, order parameters and a control parameter with well defined behaviour that lend themselves to analytical study, yet prove to be easy to calculate or estimate in large systems.

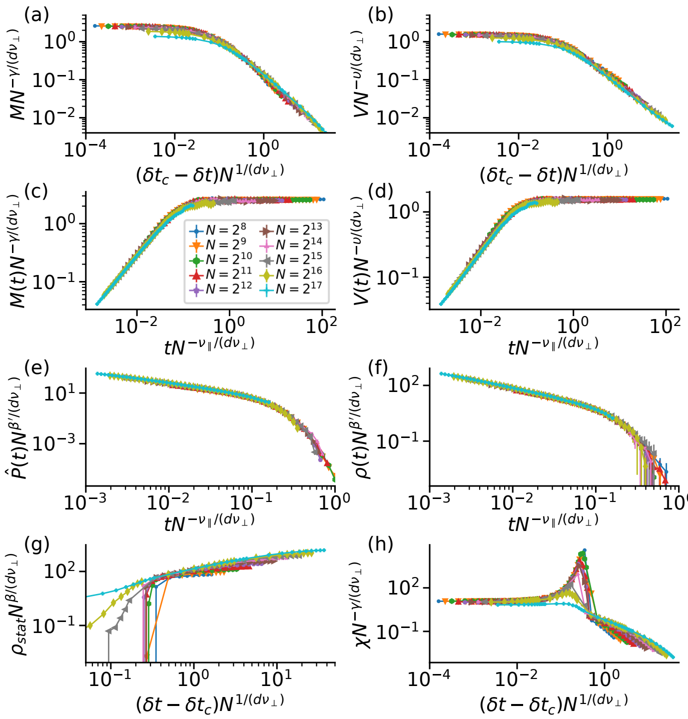

Fig. 1. An example of finite-size scaling data collapse for a temporal network constructed from random 9-regular networks with a Poisson processes ($\lambda = 1$) link activations. The system sizes, measured in number of nodes, are shown in the legend. The collapse of the trajectories confirm that the behaviour of the system can be explained using the mean-field exponents of directed percolation $\beta = \beta' = \nu_{\parallel} = 1$ and $\nu_{\perp} = 1/2$. Plots display collapse of (a,c) mean cluster mass M, (b,d) volume V, and (e) survival probability $\hat{P}(t)$ for single-source spreading scenarios. (f) Particle density $\rho(t)$, (g) static density ρ_{stat} and (h) susceptibility $\chi(\delta t, 0)$ as a function of δt for the homogeneous, fully-occupied initial condition. This is similarly verified on a verity of systems with different levels of spatial and temporal inhomogeneity [2].

References

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