Perturbative methods for estimating relative contributions to network reliability

Stephen Eubank and Madhurima Nath

Consider a discrete-time Markov process for the system configuration of a probabilistic graph dynamical system defined by $\{G, D\}$, i.e., the states of each node in a finite, node- and/or edge-weighted and directed graph G at time $t+1$ depend probabilistically on the states of their neighbors at time t according to a local function D. We are interested in the probability that introducing a perturbation to the state of a single node i at time t will affect the state of a different node j at time $t + \Delta$. The dependence of this probability on G was codified by Moore and Shannon in *network reliability* theory. It is equivalent to probabilistic satisfiability and, hence, computing or even approximating it are computationally hard. Nevertheless, the dependence of the probability on D is the subject of statistical physics, where perturbative methods have been used to estimate answers with the highest precision of any physical measurement. We explore this seeming contradiction and combine the two perspectives in a way that satisfies important physical constraints, and indicate how to use the combination to determine to a good approximation the influence any single microscopic interaction has on specific macroscopic behaviors of the whole system.

This framework is often used to study cascading failures or infectious disease epidemiology in the absence of re-infection using a simple diffusive process, the SIR compartmental model of independent Poisson processes with rates determined by edge weights and an overall transmissibility x. The Moore-Shannon network reliability, $R(G, x)$, denotes the probability that a perturbation at node i ever arrives at node j. For integer rates, the reliability is polynomial in $1 - x$ and its highest degree is the sum of weights on all the edges. Although the probability of transmission along any simple path can be evaluated easily, evaluation of the total probability is complicated by the possibility (near certainty) that G contains multiple, overlapping paths. The *disjunction* among the different paths is a major source of computational complexity. Our method not only *allows* for this possibility, but shows exactly how the multiplicity and overlaps affect the result. It can be used for any monotonic system, i.e., one in which, if subgraphs $g_1 \subseteq g_2 \subseteq G$, $R(g_1, x) \leq R(g_2, x)$. In this case, we can easily expand the disjunction using the Inclusion-Exclusion relationship, and introduce an approximation by truncating the expansion at depth D, including all combinations of no more than D minimal paths. This produces a Taylor series approximation for $R(G, x)$ at $x = 1$. Since $R(G, x)$ is the two-point correlation function in statistical physics, this approximation is exactly the strong coupling perturbative expansion of statistical physics. The paths from nodes i to j are the corresponding Feynman diagrams.

There is an appealing symmetry between the weak- and strong-coupling expansions and the Min Cut /Max Flow relationship. It is possible to construct both the weak- and strong-coupling approach using this method, and to interpolate between them in a way that respects unitarity and positivity. Writing the reliability in terms of a disjunction over minimal cut sets, as opposed to minimal paths, and expanding this disjunction as above provides a Taylor series approximation to the reliability at $x = 0$. Furthermore, a change of basis from the power basis to the Bernstein basis allows us to relate the coefficients of the two expansions and easily develop interpolations that respect positivity and unitarity. For complete details, including examples, see [1].

These methods allow us to ignore "background" reliability of a network and sensitively distinguish the contributions of different edges to overall reliability. Our recent work [2] shows the results of reducing the probability of cascading failure by removing the most influential edge using this method on a commodity trade network, and the results are compared with a well-known heuristic.

References

[1] S. Eubank, M. Nath, Y. Ren, and A. Adiga. Perturbative methods for mostly monotonic probabilistic satisfiability problems. *arXiv preprint arXiv:2206.03550*, 2022.

[2] R. Mishra et al. Community detection using Moore-Shannon network reliability: Application to food networks. *Complex Networks 2022*, (accepted).